
Expectation and Midterm 1

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Probability and Statistics for Engineers Fall 2026

Monday, October 12	Expectation, linearity, and indicators
Wednesday, October 14	Review through the binomial distribution
Friday, October 16	Midterm 1

Course support and deadlines

Ed Discussion: questions, announcements, and private messages. **No Canvas Inbox.**

[Open Ed from Canvas](#) All times Pacific

[Full contact details and meeting links](#)

Instructor / TAs	Office hours	Optional, ungraded discussions
Antonio de Leon	Fri 3–4 p.m. (in person) GSC Room 204	—
Jeannine Romero	Tue / Thu 10–11 a.m.	01A: Fri 9:20–10:25 a.m. R Carson Acad 240 (in person)
Ehsan Mirafzali	Mon 3–5 p.m.	01B: Fri 10:40–11:45 a.m. (Zoom)
Seokjun Choi	Wed 3–4 p.m. 5:05–6:05 p.m.	01C: Wed 4–5:05 p.m. (Zoom)

Upcoming deadlines

HW and quizzes: **10 p.m. Pacific**

HW 2	Mon, Oct 12	Gradescope
Quiz 3	Wed, Oct 14	Canvas
Midterm 1	Fri, Oct 16	During class; Classroom Unit 002

Midterm 1 covers required material through Oct 9; expectation is excluded.

HW: 24-hour grace period. Quizzes: no grace period. Approved accommodations apply.



From a distribution to its mean

Today: summarize a distribution by its mean, then use expectation rules to simplify calculations.

Three independent fair coin tosses. Let X be the number of heads.

Value x	0	1	2	3
$p_X(x) = P(X = x)$	1/8	3/8	3/8	1/8

Support: $S_X = \{0, 1, 2, 3\}$, the values with positive probability.

Question: over many repetitions, what average head count should we expect?

Midterm 1 covers required material through October 9. Expectation, variance, and joint distributions are excluded.

Expected value: a weighted mean

Let X have finite support S_X and PMF $p_X(x) = P(X = x)$.

Expected value, or mean

$$\mu = \mathbb{E}[X] = \sum_{x \in S_X} x p_X(x).$$

- 1 List the values in the support.
- 2 Multiply each value by its probability.
- 3 Add the weighted values.

The probabilities are nonnegative and add to one.

$\mathbb{E}[X]$ is a fixed number, measured in the same units as X .

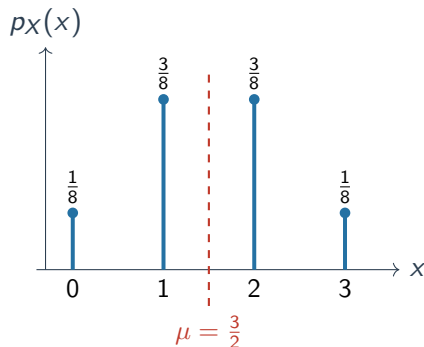
All examples today have finite support, so their expectations are finite.

A mean need not be a possible value

Number of heads in three fair tosses

$$\begin{aligned}\mathbb{E}[X] &= 0\frac{1}{8} + 1\frac{3}{8} + 2\frac{3}{8} + 3\frac{1}{8} \\ &= \frac{12}{8} = \frac{3}{2} \text{ heads.}\end{aligned}$$

The mean is 1.5, even though X only takes the values 0, 1, 2, 3.



Quick check. $Y = 0$ with probability $3/4$, or $Y = 2$ with probability $1/4$. Find $\mathbb{E}[Y]$.

Over many independent repetitions, the observed average tends toward the model mean.

Constants, scale, and shift

For constants a, b, c and a random variable X with finite expectation:

Constant $\mathbb{E}[c] = c$

Scale and shift $\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$

Scaling the values scales the mean. Adding a constant shifts the mean.

Example. In three fair tosses, award $+1$ for each head and -1 for each tail. If X counts heads, the score is

$$Z = X - (3 - X) = 2X - 3.$$

Find $\mathbb{E}[Z]$ using $\mathbb{E}[X] = 3/2$.

Linearity of expectation

For random variables with finite expectations and constants a, b :

A weighted sum

$$\mathbb{E}[aX + bY] = a\mathbb{E}[X] + b\mathbb{E}[Y].$$

Independence is not required.

Example. Let X count heads and Y count tails in three fair tosses. Knowing X determines $Y = 3 - X$, so they are dependent. Yet

$$\mathbb{E}[X + Y] = \mathbb{E}[3] = 3 = \frac{3}{2} + \frac{3}{2} = \mathbb{E}[X] + \mathbb{E}[Y].$$

The same rule applies to any finite sum:

$$\mathbb{E}\left[\sum_{i=1}^n a_i X_i\right] = \sum_{i=1}^n a_i \mathbb{E}[X_i].$$

Nonnegativity, order, and bounds

Assume the expectations are finite.

Property	Condition	Conclusion
Nonnegativity	$X \geq 0$	$\mathbb{E}[X] \geq 0$
Monotonicity	$X \leq Y$	$\mathbb{E}[X] \leq \mathbb{E}[Y]$
Bounds	$a \leq X \leq b$	$a \leq \mathbb{E}[X] \leq b$

The inequalities in the conditions must hold with probability one.

Example. Let X count heads in three tosses and cap a reward at two points: $Y = \min(X, 2)$. Since $0 \leq Y \leq X$, what can we say about $\mathbb{E}[Y]$?

Expectation of a function

For a real-valued function g and finite support S_X :

$$\mathbb{E}[g(X)] = \sum_{x \in S_X} g(x) p_X(x).$$

Transform each value, then weight it by its original probability.

x	0	1	2	3
$p_X(x)$	1/8	3/8	3/8	1/8
$g(x) = x^2$	0	1	4	9

$$\mathbb{E}[X^2] = 0 \frac{1}{8} + 1 \frac{3}{8} + 4 \frac{3}{8} + 9 \frac{1}{8} = 3.$$

In general, $\mathbb{E}[g(X)] \neq g(\mathbb{E}[X])$. Here $\mathbb{E}[X^2] = 3$, while $(\mathbb{E}[X])^2 = 9/4$. $\mathbb{E}[X^2]$ is the second moment. We will use it when studying variance.

Mean of an indicator

An indicator turns a yes/no event into a numerical random variable:

$$I_A = \begin{cases} 1, & \text{if event } A \text{ occurs,} \\ 0, & \text{otherwise.} \end{cases}$$

$$\begin{array}{c|cc} \text{Value } u & 0 & 1 \\ p_{I_A}(u) & 1 - P(A) & P(A) \end{array}$$

The same weighted-mean rule gives

$$\mathbb{E}[I_A] = 0[1 - P(A)] + 1 P(A) = P(A).$$

Check: a request fails with probability 0.02. What is the mean of its failure indicator?

For $I_A \sim \text{Bern}(p)$, $\mathbb{E}[I_A] = p$, also at $p = 0, 1$ (zero-mass terms vanish).

Mean of a count

Let I_i be 1 when trial i is a success and 0 otherwise.

$$N = I_1 + \cdots + I_n.$$

For example, the pattern success, failure, success gives $N = 1 + 0 + 1 = 2$.

$$\mathbb{E}[N] = \sum_{i=1}^n \mathbb{E}[I_i] = \sum_{i=1}^n P(\text{success on trial } i).$$

When every success probability is p

$$\mathbb{E}[N] = np.$$

If the trials are also mutually independent, $N \sim \text{Bin}(n, p)$.

Check: for the three fair tosses, $\mathbb{E}[X] = 3(1/2) = 3/2$ again.

Independence is needed to justify the binomial model, not this mean calculation.

Sampling without replacement

A bag has three red balls and two blue balls. Draw two uniformly without replacement. Let N count reds; I_i indicates that draw i is red.

- 1 Express N using indicators and find $\mathbb{E}[N]$.
- 2 Show that the draws are dependent. Why does your mean calculation still work?

Distinguish the unconditional chance of red from the chance after seeing the first draw.

Expectation: the main tools

Task	Rule
Average a function	$\mathbb{E}[g(X)] = \sum_{x \in S_X} g(x)p_X(x)$
Scale and shift	$\mathbb{E}[aX + b] = a\mathbb{E}[X] + b$
Add variables	$\mathbb{E}[\sum_i a_i X_i] = \sum_i a_i \mathbb{E}[X_i]$
Bound a mean	$a \leq X \leq b \implies a \leq \mathbb{E}[X] \leq b$
Count events	$\mathbb{E}[\sum_i I_{A_i}] = \sum_i P(A_i)$

Exit check: which rule gives $\mathbb{E}[X]$? Does linearity require independence? Must $\mathbb{E}[X^2] = (\mathbb{E}[X])^2$?

Expectation is outside Midterm 1. Next: variance and further uses of moments.

Midterm 1 review

Scope: events and counting; conditioning, total probability and Bayes; independence; random variables, PMFs/CDFs; Bernoulli and binomial probabilities.

Excluded

Expectation, variance, and joint distributions. Monday's lesson is outside Midterm 1.

We will work selected questions from the released practice paper.

Before each calculation: identify the random experiment, the requested event, and the assumptions.

The practice paper illustrates reasoning and answer detail. It does not predict which topics or question forms will appear on the exam.

Allowed: one page of notes (a cheat sheet) and a basic nonprogrammable calculator.

Concept checks

Released practice: Question 1(a), (b), (d), (f). Decide first; then give a short correction if false.

- ① $P(A) = 2/5$ and $P(B) = 3/5$ imply that A and B are disjoint.
 - ② $P(A \cap B) = P(A)P(B)$ implies that A and B are independent.
 - ③ A finite PMF is valid whenever every listed probability is between 0 and 1.
 - ④ Three favorable outcomes out of twelve always give probability $1/4$.
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One valid counterexample disproves a universal claim.

Counting codes and conditioning

Released practice: Question 3

Three-letter codes use A, B, C, D, E . Repetition is allowed; all 5^3 codes are equally likely.

(a) Exactly two different letters

How many codes?

(b) Different endpoints

Given this condition, find the probability of exactly two different letters.

A count need not be a probability. A condition changes which outcomes are being counted.

Recovering probabilities from conditional information

Released practice: Question 4(a)–(b)

E : shipped express. D : delivered the next day.

$$P(E) = \frac{2}{5}, \quad P(D | E) = \frac{3}{5}, \quad P(E | D) = \frac{4}{5}.$$

Find $P(D)$ **and** $P(D | E^c)$. Start with the shared intersection.

$P(E | D)$ and $P(D | E)$ use different denominators.

An event table: exactly one and independence

Released practice: Question 4(c)–(d). The table organizes the probabilities just found.

	D	D^c	Total
E	12/50	8/50	20/50
E^c	3/50	27/50	30/50
Total	15/50	35/50	1

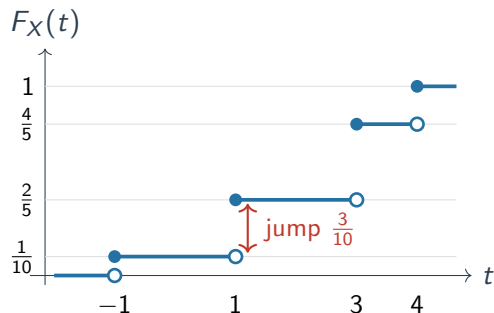
(c) Find the probability that exactly one of E, D occurs.

(d) Are E and D independent? Give a numerical check.

Exactly one excludes the intersection; at least one includes it.

CDF jumps and endpoint probabilities

Released practice: Question 2. This plot is the given piecewise CDF.



(a) Give the support and PMF.

(b) Find $P(X < 1)$ and $P(-1 < X \leq 3)$.

$F_X(t) = P(X \leq t)$. A filled dot gives the value at a jump.

$$p_X(x) = F_X(x) - F_X(x^-), \quad P(a < X \leq b) = F_X(b) - F_X(a).$$

$F_X(x^-)$: the level just left of x . Here $P(X \leq 1) = 2/5$ but $P(X < 1) = 1/10$.

A binomial model and a tail event

Released practice: Question 5(a)

Six independent sensor readings; each is incorrect with probability $1/5$. Let N count incorrect readings.

State the distribution and support; find $P(N \geq 5)$.

The coefficient chooses the positions of the five incorrect readings.

Model check: fixed number of trials, two outcomes per trial, common probability, mutual independence.

Conditioning on the total number of errors

Released practice: Question 5(b). Keep the six independent readings with error probability $1/5$.

Given $N = 2$, find the probability that the first two readings are both correct.

Position	1	2	3	4	5	6
Required	correct	correct	the two errors must occur here			

Why are the possible pairs of error positions equally likely?

$(4/5)^2$ answers the unconditional question. Conditioning on the total changes the probabilities.

When the binomial model no longer applies

Released practice: Question 5(c)

For this part only, the sixth reading has error probability $1/2$. The first five still have error probability $1/5$; all six are independent.

Find the probability that all six are correct. Is $N \sim \text{Bin}(6, 1/5)$ still valid?

Independence and equal success probabilities are separate assumptions.

Related event: at least one error is the complement of all six correct.

Writing a complete solution

Problem type	What to show
Counting	Explain each choice; show why each outcome is counted once.
Conditional probability	Identify the conditioning event and its denominator.
PMF / CDF	State the support and zero elsewhere; check endpoint inclusion.
Binomial	Give n , p , support, assumptions, and the requested event.
Independence	Use a product or conditional-probability comparison.

Exact expressions may be left unevaluated. Label each part and include the reasoning that leads to your answer.

Use all required course topics when studying. This review is a sample of methods, not an exam blueprint.

Midterm 1

Examination meeting

Friday, October 16, 2026.

55 minutes of working time within the scheduled class period.

Scope

Required material through October 9: events, counting, conditioning, independence, random variables, PMFs/CDFs, and Bernoulli/binomial laws.

Excluded

Expectation, variance, and joint distributions.

Materials

One page of notes (a cheat sheet) and a basic nonprogrammable calculator. No communication or other electronic resources.

Approved accommodations are handled separately. Follow the examination's written instructions.

Optional practice: a nonlinear cost

A daily fault count X has the following distribution. A cost in dollars is $C = 10X^2$.

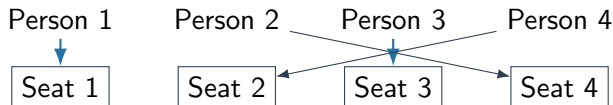
x	0	1	2
$p_X(x)$	1/2	3/10	1/5
$10x^2$	0	10	40

Find $\mathbb{E}[C]$. Compare it with $10(\mathbb{E}[X])^2$.

Use $\mathbb{E}[g(X)] = \sum_{x \in \mathcal{S}_X} g(x)p_X(x)$ with $g(x) = 10x^2$.

Optional: random matching

There are $n \geq 1$ labeled seats and n people, each with an assigned seat. Choose one of the $n!$ seating permutations uniformly. Let M count correct matches.

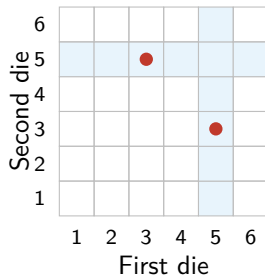


Let I_i indicate that person i gets seat i . Find $\mathbb{E}[M]$.

The diagram shows one seating of four people. The mean calculation covers all permutations.

Optional: changed conditioning

Roll two independent fair dice. Let S be their sum and B the event that at least one die shows 5.



Compare three observations

- ① At least one die shows 5.
- ② The first die shows 5.
- ③ Exactly one die shows 5.

What is the probability of $S = 8$ under each observation?

Count favorable outcomes inside the conditioning event.

References for this week

DeGroot and Schervish

Probability and Statistics, fourth edition.

- §4.1, pp. 207–209: discrete expectation and Bernoulli example.
- §4.2, pp. 217–221: affine changes, linearity, and sampling means.

Mor Harchol-Balter

Introduction to Probability for Computing, supplied 2024 version.

- §4.1, pp. 58–61: expectation and functions of a discrete variable.
- §4.2, pp. 63–64: linearity and the binomial mean.
- Example 4.12, pp. 64–65: optional random matching.

Selections only. Review: current Weeks 1–2 slides and the released Midterm 1 Practice and Solutions (October 8 revision).

Additional source: Sheng Jiang, STAT 131 Notes (July 18, 2024), pp. 48–51; Antonio de Leon's earlier STAT 131 materials.