
Bounds, the weak law, and review

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Probability and Statistics for Engineers Fall 2026

Monday, November 9

Markov, Chebyshev, and the finite-variance weak LLN

Wednesday, November 11

No lecture

Friday, November 13

Midterm 2 review

Course support and deadlines

Ed Discussion: questions, announcements, and private messages. **No Canvas Inbox.**

[Open Ed from Canvas](#) All times Pacific

[Full contact details and meeting links](#)

Instructor / TAs	Office hours	Optional, ungraded discussions
Antonio de Leon	Fri 3–4 p.m. (in person) GSC Room 204	—
Jeannine Romero	Tue / Thu 10–11 a.m.	01A: Fri 9:20–10:25 a.m. R Carson Acad 240 (in person)
Ehsan Mirafzali	Mon 3–5 p.m.	01B: Fri 10:40–11:45 a.m. (Zoom)
Seokjun Choi	Wed 3–4 p.m. 5:05–6:05 p.m.	01C: Wed 4–5:05 p.m. (Zoom)

Upcoming deadlines

HW and quizzes: **10 p.m. Pacific**

HW 5	Thu, Nov 12	Gradescope
Quiz 7	Thu, Nov 12	Canvas
Midterm 2	Mon, Nov 16	During class; Classroom Unit 002

No lecture Wed, Nov 11. [Check Ed for holiday changes to support sessions.](#)

HW: 24-hour grace period. Quizzes: no grace period. Approved accommodations apply.



Bounds from partial information

A hypothetical delay W , measured in minutes, satisfies

$$W \geq 0, \quad \mathbb{E}[W] = 6, \quad \text{Var}(W) = 9.$$

What can we say about $P(W \geq 18)$ without a full probability law?

A tail probability measures how often a threshold is exceeded.

Markov's inequality

If $Z \geq 0$ almost surely, $\mathbb{E}[Z] < \infty$, and $a > 0$, then

$$P(Z \geq a) \leq \frac{\mathbb{E}[Z]}{a}.$$

Every outcome in the tail contributes at least a to Z .

Derive the bound by comparing Z with $a \mathbf{1}_{\{Z \geq a\}}$.

A useful probability bound is $\min\{1, \mathbb{E}[Z]/a\}$.

Chebyshev's inequality

Suppose $\mathbb{E}[X] = \mu$ and $\text{Var}(X) = \sigma^2 < \infty$. For $a > 0$,

$$P(|X - \mu| \geq a) \leq \frac{\sigma^2}{a^2}.$$

Squared distance from the mean is a nonnegative random variable.

Apply Markov to $Z = (X - \mu)^2$. What is the new threshold?

Use $\min\{1, \sigma^2/a^2\}$ as an upper bound. Finite variance is essential here.

One-sided and two-sided events

For the delay, $\mathbb{E}[W] = 6$, $\text{Var}(W) = 9$, and $W \geq 0$.



$$\{W \geq 18\} \subseteq \{|W - 6| \geq 12\}.$$

Event inclusion justifies the one-sided bound. There is no factor of $1/2$.

Mean and variance of an average

Let $X_1, \dots, X_n$ be iid with mean μ and finite variance σ^2 .

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i.$$

Derive its mean and variance. Identify the step that uses independence.

Linearity of expectation holds even when the observations are dependent.

A finite-sample guarantee

lid measurements have mean μ and variance at most 4 units². How large should n be to guarantee

$$P(|\bar{X}_n - \mu| < 1/2) \geq 0.95?$$

In general: if $\sigma^2 \leq v$, $0 < \delta < 1$, and $\varepsilon > 0$, use $n \geq \max\{1, \lceil v/(\delta\varepsilon^2) \rceil\}$.

This certifies a sufficient sample size. The true distribution may need fewer observations.

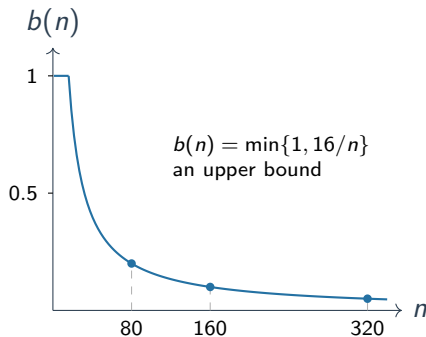
Convergence in probability

For every fixed $\varepsilon > 0$,

$$P(|Z_n - b| \geq \varepsilon) \longrightarrow 0.$$

We write $Z_n \xrightarrow{P} b$.

Fix the tolerated error first. Then let the sample size grow.



The curve connects upper bounds at positive integer sample sizes. It is not the actual tail probability.

Weak LLN: statement and proof

Let $X_1, X_2, \dots$ be iid with mean μ and finite variance σ^2 . Then

$$\overline{X}_n \xrightarrow{P} \mu.$$

Use the variance of the average in Chebyshev's inequality.

This proof establishes the finite-variance weak law. It does not assume a normal distribution.

Why dependence matters

Let $P(Z = -1) = P(Z = 1) = 1/2$ and set $X_i = Z$ for every i . Each observation has mean 0 and variance 1.

$$\bar{X}_n = \frac{1}{n} \sum_{i=1}^n Z.$$

Identical marginal distributions alone do not give the iid variance calculation.

Hypotheses and interpretation

An iid average has variance at most $4/n$. At $n = 320$, Chebyshev gives

$$P(|\bar{X}_n - \mu| \geq 1/2) \leq 0.05.$$

State the corresponding success guarantee. Does the probability equal 0.95?

The weak LLN concerns a probability limit, not a monotone path toward μ .

Midterm 2: scope and methods

Required material through November 9; proposed exam November 16. Working time: 55 minutes.

Information	Method and conclusion
A probability law	Sum or integrate over the event; obtain an exact value.
Mean and variance	Chebyshev gives a bound when its hypotheses hold.
iid observations	The finite-variance weak LLN gives convergence in probability.

Expectation, variance, joint/conditional laws, covariance, and elementary CDF transformations remain in scope.

CLT, inference, and stochastic processes are excluded.

Expectation without independence

Twelve components each have failure probability 0.15. Let I_j indicate failure of component j , and let $N = \sum_{j=1}^{12} I_j$. Find $\mathbb{E}[N]$. Can we conclude that N is binomial?

The method for a mean may need fewer assumptions than the full distribution.

A joint probability table

$p_{X,Y}(x,y)$	$y = 0$	$y = 1$	$p_X(x)$
$x = 0$	3/8	1/8	
$x = 1$	1/8	3/8	
$p_Y(y)$			1

Find the marginals and $P(Y = 1 \mid X = 1)$. Are X and Y independent?

Normalize the row selected by the conditioning event.

Covariance in a linear combination

For the preceding table:

$$\mathbb{E}[X] = \mathbb{E}[Y] = \frac{1}{2}, \quad \text{Var}(X) = \text{Var}(Y) = \frac{1}{4}, \quad \mathbb{E}[XY] = \frac{3}{8}.$$

Find $\text{Cov}(X, Y)$, $\rho_{X,Y}$, and $\text{Var}(2X - Y)$.

A negative coefficient changes the sign of the covariance contribution.

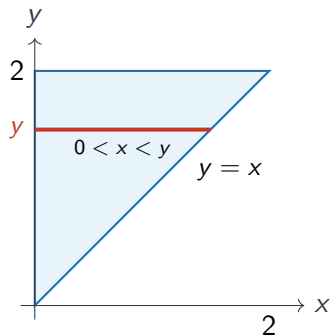
Exact exponential and normal probabilities

For $T \sim \text{Exp}(2)$, where 2 is a rate, find $P(T > 2 \mid T > 1)$.

For $G \sim N(10, 4)$, find $P(8 < G < 12)$ in terms of Φ .

The stated normal model makes the second calculation exact.

Probability over a triangular support



$$f_{X,Y}(x,y) = \frac{1}{2} I_{\{0 < x < y < 2\}}$$

Derive $f_Y(y)$ and find $P(Y < 1)$.

The triangle has area 2. Its density $1/2$ therefore integrates to one.

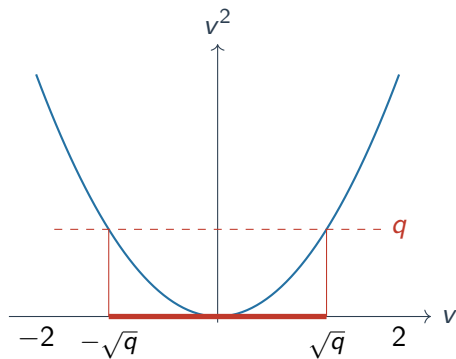
A conditional slice

In the same triangle, $f_Y(y) = \frac{y}{2} I_{\{0 < y < 2\}}$. Find the conditional density of X given $Y = y$, then compute

$$P(X < 1 \mid Y = 3/2), \quad \mathbb{E}[X \mid Y = 3/2].$$

The density ratio uses $f_Y(y) > 0$, not $P(Y = y)$.

A transformation with two preimages



Let $V \sim \text{Unif}(-2, 2)$ and $Q = V^2$. For $0 \leq q < 4$,

$$\{Q \leq q\} = \{-\sqrt{q} \leq V \leq \sqrt{q}\}.$$

Find the full CDF and then the density.

A bound and a limit

lid measurements have mean μ and variance at most 9.

- ① Bound $P(|\bar{X}_{100} - \mu| \geq 1)$.
- ② Find a sufficient n for failure probability at most 0.07.
- ③ Explain the limit for each fixed $\varepsilon > 0$.

Round a sufficient integer sample size upward.

Three statements to repair

- ❶ Zero covariance implies independence.
 - ❷ Differentiating a CDF always recovers its whole probability law.
 - ❸ The weak LLN makes every sample path move closer to μ at every step.
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State exactly what the result establishes.

Optional review: selecting a method

This block is ungraded and can be completed independently. Choose a method before calculating:

- ① A probability from a joint PMF, given an event in one coordinate.
 - ② $P(8 < G < 12)$ when $G \sim N(10, 4)$.
 - ③ The law of $g(X)$ when g is not one-to-one.
 - ④ A deviation probability when only mean and variance are supplied.
 - ⑤ The behavior of an iid average as n grows.
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Optional review: table and covariance

$p_{X,Y}(x,y)$	$y = 0$	$y = 1$
$x = 0$	$1/5$	$3/10$
$x = 1$	$1/10$	$2/5$

Find the marginals, $P(X = 1 \mid Y = 1)$, $\text{Cov}(X, Y)$, and $\text{Var}(X + Y)$.

Optional review: a nonuniform joint density

Let $f_{X,Y}(x,y) = \frac{2}{3}(x+2y)I_{\{0 < x < 1, 0 < y < 1\}}$. Find f_Y , $f_{X|Y}$, and $P(X < 1/2 \mid Y = 1/2)$.

A rectangular support alone does not establish independence.

Optional review: a decreasing transformation

Let $U \sim \text{Unif}(0,1)$ and $R = 1 - \sqrt{U}$. Find its full CDF, its density, and $P(R > 1/2)$.

For $0 \leq r \leq 1$, translate $R \leq r$ into an inequality in U .

Optional review: a sample-size certificate

lid observations have mean μ and variance at most 5.

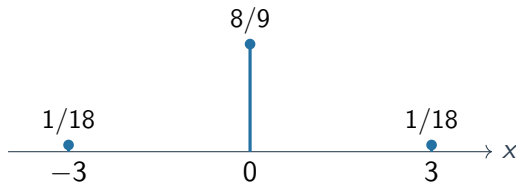
- ① At $n = 80$, bound $P(|\bar{X}_n - \mu| \geq 1/2)$.
- ② Find a sufficient integer n to make that probability at most 0.1.
- ③ Write the resulting success guarantee.

Optional review: the weak-LLN proof

State the finite-variance iid weak LLN and reconstruct its proof. Your argument must specify a positive tolerance and the moments of the average.

Optional: equality in Chebyshev's bound

Let X take the values $-3, 0, 3$ with probabilities $1/18, 8/9, 1/18$.



Optional extension. Using a bound does not require proving that it is attained.

Optional: exact tails and bounds

Let $X \sim \text{Exp}(\lambda)$ and consider $P(X \geq a/\lambda)$, $a > 1$.

a	Exact	Markov	Chebyshev
2	e^{-2}	$1/2$	1
3	e^{-3}	$1/3$	$1/4$

Adapted from Harchol-Balter, supplied 2024 text, Exercise 18.14(a)–(c), p. 323.

Reading selections

DeGroot and Schervish, fourth edition

- p. 349: Markov and Chebyshev, Theorems 6.2.1–6.2.2 and proofs.
- p. 350: mean and variance of the average, Theorem 6.2.3.
- pp. 350–351: Example 6.2.1 only.
- p. 352: Definition 6.2.1 and Theorem 6.2.4; its proof on p. 353.

Harchol-Balter, supplied 2024 version

- pp. 98–99: Theorems 5.16–5.17 and proofs.
- Optional: p. 323, Exercise 18.13 and Exercise 18.14(a)–(c).

These are selected passages. The slides and notes supply the required statements and proofs.