STAT 131 — Discussion (Week 8)

Quick recap

Discrete transform

If $Y = r(X)$ and X is discrete, $g(y) = \Pr(Y = y) = \sum_{x:r(x)=y} p_X(x)$.

CDF method

If $Y = r(X)$ and X is continuous, the safest recipe is

$$f,\ \mathrm{Supp}(X) \longrightarrow F \longrightarrow \mathrm{Supp}(Y) \longrightarrow G \longrightarrow g.$$

First compute

$$F(x) = \Pr(X \le x) = \int_{-\infty}^{x} f(t)\,\mathrm{d}t.$$

Then, for $y \in \mathrm{Supp}(Y)$,

$$G(y) = \Pr(Y \le y) = \Pr(r(X) \le y), \qquad g(y) = \frac{\mathrm{d}}{\mathrm{d}y}G(y).$$

This method works even when r is not one-to-one!

One-to-one shortcut

If r is one-to-one on $\mathrm{Supp}(X)$, let $x = r^{-1}(y)$. Then

$$g(y) = f\left(r^{-1}(y)\right)\left|\frac{\mathrm{d}}{\mathrm{d}y}r^{-1}(y)\right|, \qquad y \in \mathrm{Supp}(Y).$$

Always check the support before using the shortcut.

Sum convolution

If $S = X_1 + X_2$ and X_1, X_2 are independent continuous random variables, then

$$f_S(s) = \int_{-\infty}^{\infty} f_{X_1}(x)\,f_{X_2}(s - x)\,\mathrm{d}x.$$

The actual limits come from the supports of X_1 and X_2.

Difference convolution

If $D = X_1 - X_2$ and X_1, X_2 are independent continuous random variables, then

$$f_D(d) = \int_{-\infty}^{\infty} f_{X_1}(d + u)\,f_{X_2}(u)\,\mathrm{d}u.$$

The support determines the valid integration limits.

Maximum of i.i.d. sample

Let $Y_n = \max\{X_1, \ldots, X_n\}$, where the X_i are i.i.d. with CDF F and density f. Then

$$G_n(y) = \Pr(Y_n \le y) = [F(y)]^n,$$

and, where differentiable,

$$g_n(y) = n[F(y)]^{n-1} f(y).$$

Use the event "all observations are $\le y$."

Minimum of i.i.d. sample

Let $Y_1 = \min\{X_1, \ldots, X_n\}$. Then

$$G_1(y) = 1 - \Pr(Y_1 > y) = 1 - [1 - F(y)]^n,$$

and, where differentiable,

$$g_1(y) = n[1 - F(y)]^{n-1} f(y).$$

Use the complement: "not all observations are $> y$."

Version 1

Problem 1 Maximum of exponential random variables

Let
$$X_1, X_2 \overset{\text{iid}}{\sim} \text{Exp}(2), \qquad Y_2 = \max\{X_1, X_2\}.$$
Find the CDF and density of Y_2.

Problem 2 Convolution

Let
$$X_1, X_2 \overset{\text{iid}}{\sim} \text{Exp}(2), \qquad D = X_1 - X_2.$$
Find the support and density of D.

Version 2

Problem 1 Maximum of exponential random variables

Let

$$X_1, X_2 \stackrel{\text{iid}}{\sim} \text{Exp}(3), \qquad Y_1 = \min\{X_1, X_2\}.$$

Find the CDF and density of Y_1.

Problem 2 Convolution

Let

$$X_1, X_2 \stackrel{\text{iid}}{\sim} \text{Exp}(3), \qquad D = X_1 - X_2.$$

Find the support and density of D.