STAT 131 – Lecture Review Week 5

Discrete distributions, continuous densities, CDFs, and quantiles

Antonio Aguirre

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Based on Lectures 4.1 and 4.2, with midterm-style examples.

How to use this review

What this hour is for

This is not a second lecture. The goal is to connect formulas to the words in a problem: **what is being counted, what is random, and how probabilities are computed**.

Four questions to ask every time

1. Is X **discrete** or **continuous**?
2. If X is discrete, what values can it take and what is $\Pr(X = x)$?
3. If X is continuous, what area under the density represents the probability?
4. If a c.d.f. is involved, are we accumulating mass or area up to x?

Main habit

Define the random variable before choosing a formula: $X =$ what number?

Big picture: from outcomes to distributions

Random variable

A random variable is a numerical summary of an outcome:

$$X : S \to \mathcal{D} \subseteq \mathbb{R}.$$

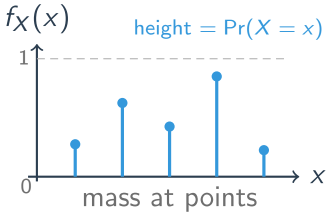
The distribution tells us how probability is assigned to values of X.

Do not skip the support

The support $\mathcal{D}$ is the set of possible values. Many mistakes come from summing or integrating over the wrong set.

Two probability languages

Discrete	probabilities are **masses**: $f_X(x) = \Pr(X = x)$
Continuous	probabilities are **areas**: $\Pr(a \le X \le b) = \int_a^b f_X(x)\,\mathrm{d}x$



The distribution chooser

Most of the work is recognizing the experiment

Story	**Random variable**	**Model**
All k values equally likely	selected value	$\mathrm{Unif}\{1, \ldots, k\}$
One yes/no trial	success indicator	$\mathrm{Bern}(p)$
Fixed n independent yes/no trials	number of successes	$\mathrm{Binom}(n, p)$
Draw n without replacement	number of red/defective items	$\mathrm{Hypergeom}(n, A, B)$
Repeat until first success	failures before success	$\mathrm{Geom}(p)$
Repeat until r successes	failures before r successes	$\mathrm{Neg\text{-}Bin}(r, p)$
Events in time or space at a rate	number of events	$\mathrm{Poisson}(\lambda)$

Reading tip

Independent with replacement points toward binomial. Without replacement points toward hypergeometric.

P.m.f. rules: small but important

Discrete probability mass function

For a discrete random variable X with support $\mathcal{D}$,

$$f_X(x) = \Pr(X = x), \qquad x \in \mathcal{D}.$$

A valid p.m.f. must satisfy

$$0 \leq f_X(x) \leq 1, \qquad \sum_{x \in \mathcal{D}} f_X(x) = 1.$$

Probability of a set of values

If $A \subseteq \mathcal{D}$, then

$$\Pr(X \in A) = \sum_{x \in A} f_X(x).$$

Binomial: fixed number of independent trials

Setup

- n independent Bernoulli trials.
- Each trial has success probability p.
- X = number of successes.

$$X \sim \mathrm{Binom}(n, p), \qquad x = 0, 1, \ldots, n.$$

Formula

$$\Pr(X = x) = \binom{n}{x} p^x (1-p)^{n-x}.$$

Where the pieces come from

$$\underbrace{\binom{n}{x}}_{\text{locations of successes}} \quad \underbrace{p^x}_{\text{successes}} \; \underbrace{(1-p)^{n-x}}_{\text{failures}}$$

Typical wording: "10 seeds, each germinates independently with probability 0.7."

Hypergeometric: sampling without replacement

Setup

A box has A red items and B blue items. Draw n items **without replacement**. Let

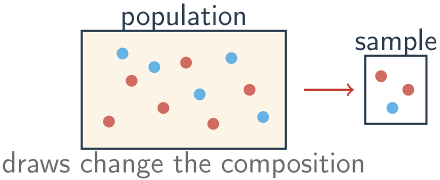
$$X = \text{number of red items drawn.}$$

Then

$$X \sim \mathrm{Hypergeom}(n, A, B).$$

Formula

$$\Pr(X = x) = \frac{\binom{A}{x}\binom{B}{n-x}}{\binom{A+B}{n}}.$$



Reading cue

No replacement means the trials are not independent.

Midterm-style example: red and blue balls

A box has 7 red and 5 blue balls. Draw 4 without replacement. Let

$$R = \text{number of red balls in the sample.}$$

Step 1: identify the model

$$R \sim \mathrm{Hypergeom}(n = 4, A = 7, B = 5).$$

We are drawing without replacement, and R counts red balls.

Step 2: compute $\Pr(R = 3)$

$$\Pr(R = 3) = \frac{\binom{7}{3}\binom{5}{1}}{\binom{12}{4}} = \frac{35 \cdot 5}{495} = \frac{35}{99}.$$

Setup matters more than arithmetic

The denominator counts all samples of size 4. The numerator counts samples with exactly 3 red and 1 blue.

Same example: add a condition

Let

$$B = \{\text{at least one blue ball is drawn}\}.$$

Find $\Pr(R = 3 \mid B)$.

Use the conditional probability definition

If $R = 3$, then exactly one blue was drawn, so $\{R = 3\} \subset B$.

$$\Pr(R = 3 \mid B) = \frac{\Pr(R = 3)}{\Pr(B)}.$$

Compute the denominator by complement

$$\Pr(B) = 1 - \Pr(\text{all 4 are red}) = 1 - \frac{\binom{7}{4}}{\binom{12}{4}} = \frac{92}{99}.$$

Thus

$$\Pr(R = 3 \mid B) = \frac{35/99}{92/99} = \frac{35}{92}.$$

Waiting-time distributions: geometric and negative binomial

Geometric

Repeat independent Bernoulli trials until the first success. Let

$$X = \text{failures before the first success.}$$

Then

$$X \sim \mathrm{Geom}(p),$$

$$\Pr(X = x) = p(1-p)^x, \quad x = 0, 1, 2, \ldots$$

Negative binomial

Repeat until the rth success. Let

$$X = \text{failures before the } r\text{th success.}$$

Then

$$\Pr(X = x) = \binom{x + r - 1}{x} p^r (1-p)^x.$$

count failures before successes

Poisson: counts in time or space

Setup

Use a Poisson model when X counts events in a fixed interval of time or space, with a roughly constant rate λ.

$$X \sim \mathrm{Poisson}(\lambda)$$

$$\Pr(X = x) = e^{-\lambda}\frac{\lambda^x}{x!}, \qquad x = 0, 1, 2, \ldots$$

Example: requests to a server

Suppose a small service receives an average of 3 requests per minute. Let

$$X = \text{requests in one minute.}$$

Then $X \sim \mathrm{Poisson}(3)$ and

$$\Pr(X \geq 2) = 1 - \Pr(X = 0) - \Pr(X = 1)$$

$$= 1 - e^{-3} - 3e^{-3} = 1 - 4e^{-3}.$$

Reading tip

At least plus an infinite tail often means use the complement.

Discrete toolbox: what to check

A compact decision table

Model	**Key phrase**	**What students often miss**
Binomial	fixed n, independent trials	the $\binom{n}{x}$ counts locations of successes
Hypergeometric	without replacement	the support may be restricted by available red/blue items
Geometric	until first success	X here counts failures, so success on first trial gives $X = 0$
Negative binomial	until r successes	the last trial must be a success
Poisson	count in a fixed time/space interval	use complements for "at least" events

Exam habit

Before applying a formula, write one sentence: "Let $X = \ldots$"

Continuous random variables: probability is area

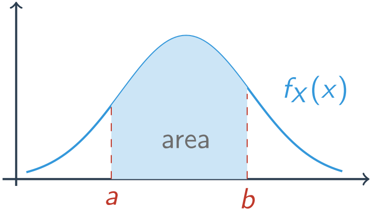
Density

A continuous random variable has a density f_X such that

$$\Pr(a \le X \le b) = \int_a^b f_X(x)\,\mathrm{d}x.$$

A valid density satisfies

$$f_X(x) \ge 0, \qquad \int_{-\infty}^{\infty} f_X(x)\,\mathrm{d}x = 1.$$



Key difference

For continuous X, $\Pr(X = x) = 0$. The density is not a point probability.

Normalizing a density

Suppose a continuous random variable has density $f_X(x) = \begin{cases} c(1-x), & 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$ Find c!

Use total area $= 1$

$$1 = \int_{-\infty}^{\infty} f_X(x)\,\mathrm{d}x = \int_0^1 c(1-x)\,\mathrm{d}x.$$

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$$1 = \int_{-\infty}^{\infty} f_X(x)\,\mathrm{d}x = \int_0^1 c(1-x)\,\mathrm{d}x.$$

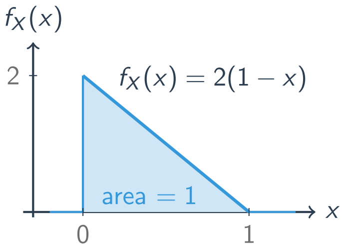
$$1 = c\left[x - \frac{x^2}{2}\right]_0^1 = \frac{c}{2}, \qquad c = 2.$$

Meaning

The density is scaled so that the entire shaded area is exactly 1.

Resulting density

$$f_X(x) = \begin{cases} 2(1-x), & 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$



From density to c.d.f.

Cumulative distribution function

For any random variable,

$$F_X(x) = \Pr(X \leq x).$$

For a continuous random variable with density f_X,

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,\mathrm{d}t.$$

When differentiable,

$$f_X(x) = F'_X(x).$$

Interpretation

The c.d.f. accumulates all probability to the left of x.

Midterm-style example: find the c.d.f.

Using the density

$$f_X(x) = 2(1-x), \qquad 0 < x < 1,$$

find $F_X(x) = \Pr(X \le x)$.

Piecewise setup

$$F_X(x) = \begin{cases} 0, & x \le 0, \\ \displaystyle\int_0^x 2(1-t)\,\mathrm{d}t, & 0 < x < 1, \\ 1, & x \ge 1. \end{cases}$$

Middle piece

$$\int_0^x 2(1-t)\,\mathrm{d}t = 2x - x^2.$$

Thus

$$F_X(x) = \begin{cases} 0, & x \le 0, \\ 2x - x^2, & 0 < x < 1, \\ 1, & x \ge 1. \end{cases}$$

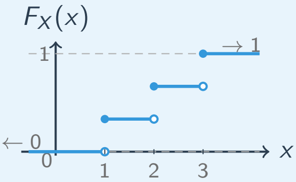
Discrete CDF vs. continuous CDF

Always true

$$F_X(x) = \Pr(X \le x), \qquad 0 \le F_X(x) \le 1, \qquad \lim_{x\to-\infty} F_X(x) = 0, \qquad \lim_{x\to\infty} F_X(x) = 1.$$

Discrete: mass accumulates by jumps

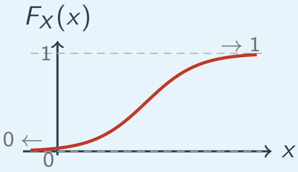
$$F_X(x) = \sum_{t \le x} f_X(t).$$



jumps; jump size $= \Pr(X = x)$

Continuous: area accumulates smoothly

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,\mathrm{d}t.$$

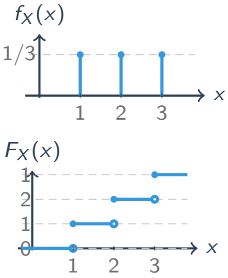


smooth accumulation; no jumps

PMF/PDF paired with the CDF

Discrete example: $X \sim \mathrm{Unif}\{1,2,3\}$

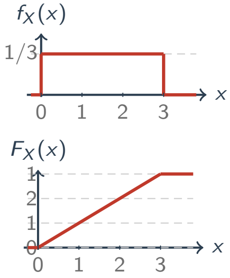
$$f_X(x) = \begin{cases} 1/3, & x \in \{1,2,3\}, \\ 0, & \text{otherwise}, \end{cases} \qquad F_X(x) = \sum_{t \le x} f_X(t).$$



CDF: adds masses, so it jumps

Continuous example: $X \sim \mathrm{Unif}[0,3]$

$$f_X(x) = \begin{cases} 1/3, & 0 \le x \le 3, \\ 0, & \text{otherwise}, \end{cases} \qquad F_X(x) = \int_{-\infty}^{x} f_X(t)\,\mathrm{d}t.$$



CDF: accumulates area smoothly

Continuous uniform: probability is length divided by total length

Model

If $X \sim \mathrm{Unif}[a, b]$, then

$$f_X(x) = \begin{cases} \dfrac{1}{b-a}, & a \le x \le b, \\ 0, & \text{otherwise.} \end{cases}$$

and

$$F_X(x) = \begin{cases} 0, & x < a, \\ \dfrac{x-a}{b-a}, & a \le x \le b, \\ 1, & x > b. \end{cases}$$

Example

A shuttle arrival is uniformly distributed between 0 and 10 minutes. What is

$$\Pr(2 \le X \le 7)?$$

Since the density is constant,

$$\Pr(2 \le X \le 7) = \frac{7-2}{10-0} = \frac{1}{2}.$$

Uniform is the continuous version of “all intervals of equal length are equally likely.”

Quantiles: reading a c.d.f. backwards

Continuous case

If F is continuous and one-to-one, the p-quantile solves

$$F(x) = p, \qquad x = F^{-1}(p).$$

For $X \sim \mathrm{Unif}[a, b]$,

$$p = \frac{x-a}{b-a} \quad \Rightarrow \quad x = a + p(b-a) = pb + (1-p)a.$$

Discrete case

For a discrete random variable, the p-quantile is the **smallest** x such that

$$F(x) \geq p.$$

Midterm-style quantile check

A discrete random variable has c.d.f. values

$$F(0) = 0.10, \qquad F(1) = 0.55, \qquad F(2) = 0.85, \qquad F(3) = 1.$$

$p = 0.50$ quantile

Find the smallest x with $F(x) \geq 0.50$.

$$F(0) = 0.10 < 0.50, \qquad F(1) = 0.55 \geq 0.50.$$

So the 0.50-quantile is $\boxed{1}$.

$p = 0.80$ quantile

Find the smallest x with $F(x) \geq 0.80$.

$$F(1) = 0.55 < 0.80, \qquad F(2) = 0.85 \geq 0.80.$$

So the 0.80-quantile is $\boxed{2}$.

Do not interpolate discrete quantiles

Discrete c.d.f.s jump. The answer is the first value where the c.d.f. reaches the target probability.

Common mistakes to avoid

Discrete mistakes

- Using a binomial model when sampling is without replacement.
- Forgetting that geometric here counts failures before success, so $X = 0$ is possible.
- Interpolating discrete quantiles instead of taking the first x with $F_X(x) \geq p$.

Continuous mistakes

- Treating $f_X(x)$ as $\Pr(X = x)$.
- Forgetting to normalize: $\int f_X(x)\,\mathrm{d}x = 1$.
- Forgetting the support when setting up an integral or c.d.f.

Reliable habit

Write the support before summing or integrating.

Final checklist

Before choosing a formula

- Define the random variable: $X =$ what number?
- Write the support: possible values or interval.
- For discrete X, use sums of p.m.f. values.
- For continuous X, use areas under a density.
- For a density with an unknown constant, normalize: $\int f = 1$.
- For a c.d.f., remember: $F(x) = \Pr(X \leq x)$.
- For quantiles, read the c.d.f. backwards.

Main message

Most errors are not algebra errors. They are setup errors: wrong model, wrong support, or wrong interpretation of $f(x)$.